UNIVERSIDADE FEDERAL DO RIO DE JANEIRO
Centro de Ciências Matemáticas e da Natureza
Instituto de Matemática

CONCURSO PARA PROFESSOR ADJUNTO A – IM
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Problem 1-1

Let A be a $n\times n$ real matrix. Then A is similar to a $n\times n$ real matrix J, where J has the following form: ~~Given an~~ $\exists$ integer k s.t. $1 \le k \le n$, J can be written as a block form

$$J = \begin{pmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_k \end{pmatrix},$$ where ~~B_k has a form~~

for each B_j, $1 \le j \le k$, $\exists\ \lambda_j \in \mathbb{R}$ s.t. B_j has the following form: a) $B_j = \begin{pmatrix} \lambda_j & 1 & & \\ & \lambda_j & 1 & \\ & & \ddots & 1 \\ & & & \lambda_j \end{pmatrix}$ or ~~b)~~

B_j could also be $[\lambda]$.

b) $B_j = \begin{pmatrix} 0 & \lambda_j \\ -\lambda_j & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & \lambda_j & 1 & & & \\ -\lambda_j & 0 & & 1 & & \\ & & 0 & \lambda_j & \ddots & \\ & & -\lambda_j & 0 & & \ddots \\ & & & & 0 & \lambda_j \\ & & & & -\lambda_j & 0 \end{pmatrix}$

Problem 1-2

~~Since~~ The roots of this characteristic polynomial

$$\det(tI - A) = t^5 + 2t^3a + a^2 t = t(t^4 + 2at^2 + a^2),$$

are $\lambda = 0, i\sqrt{5}, -i\sqrt{5}$.

The root 0 has multiplicity 1, so it must be an eigenvalue. This implies that the Jordan form of A may look like

$$J = \begin{pmatrix} 0 & -\sqrt{5} & & & \\ \sqrt{5} & 0 & & & \\ & & 0 & -\sqrt{5} & \\ & & \sqrt{5} & 0 & \\ & & & & 0 \end{pmatrix} \text{ or } J = \begin{pmatrix} 0 & -\sqrt{5} & 1 & & \\ \sqrt{5} & 0 & & 1 & \\ & & 0 & -\sqrt{5} & \\ & & \sqrt{5} & 0 & \\ & & & & 0 \end{pmatrix}$$

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Problem 1-3 From the characteristic polynomial, the Jordan normal form of B is either

$$J = I_{5\times5} \text{ or } \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & 1 & \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & & & & \\ & 1 & 1 & & \\ & & 1 & 1 & \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix}$$

$$\text{or } \begin{pmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & 1 & 1 & \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix}.$$

Recall that ~~If~~ ~~two~~ two matrices are similar $\Longleftrightarrow$ they have the same Jordan canonical form, so we must show that B and B^{-1} have the same J.C.F.

If $J = I_{5\times5}$, this is very clear (in fact, $B = B^{-1} = I_{5\times5}$).

Suppose $J = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix}$. Then $J^{-1} = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & -1 \\ & & & & 1 \end{pmatrix}$.

If we take basis $\{v_i\}_{i=1}^5$ of $\mathbb{R}^5$ to be such that $v_i = e_i$ for $i = 1, 2, 3, 4$, and $v_5 = 2e_4 + e_5$. Then $J^{-1}v_5 = \cancel{2e_4 - e_5} = 2e_4 + (e_5 - e_4) = e_4 + e_5$, so J^{-1} has the same Jordan Canonical form as J.

Apply the similar techniques for the rest of the JCF, and we're done $\blacksquare$

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Problem 2-1

Green's Theorem Let D be a ~~region~~ region bounded by a simply closed curves (which is ∂D), and ∂D is oriented positively (so that the region is always to the left). Let $F = (P, Q)$ be a C^1 function on $\mathbb{R}^2$ ($P, Q : \mathbb{R}^2 \to \mathbb{R}$).

Then $$\int_{\partial D} F \cdot T\, ds = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy.$$

Stokes' Theorem Let M be a compact surface w/ boundary, and let ∂M be oriented positively. Let $F = (P, Q, R)$ be a C^1 function on $\mathbb{R}^3$ (where $P, Q, R : \mathbb{R}^3 \to \mathbb{R}$).

Then $$\int_{\partial M} F \cdot T ds = \iint_M \operatorname{curl} F \, d\vec{n}$$

Gauss's Theorem Let V be a compact, Jordan measurable set in $\mathbb{R}^3$, and let ∂V be a surface oriented by the upper normal vector. If $F = (P, Q, R)$ is a C^1 vector field, then

$$\iint_{\partial V} F \cdot d\vec{n} = \iiint_V \operatorname{div} F \, dV.$$

Problem 2-2 Recall that $\Delta u = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) u$ and $\nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right)$. Then Gauss's theorem implies

$$\iint_{\partial R} \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right) d\vec{n} = \iiint_R \operatorname{div} \nabla u \, dx\,dy\,dz$$
$$= \iiint_R \left(\frac{\partial}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}\frac{\partial u}{\partial y} + \frac{\partial}{\partial z}\frac{\partial u}{\partial z}\right) dx\,dy\,dz$$
$$= \iiint_R \Delta u \, dx\,dy\,dz$$

Problem 2-3 Here, $\partial R = S_1 \cup S_2$, where S_i is a sphere of radius r_i, $i=1,2$. We parametrize S_i by

$$\Phi_i : [\varphi_1, \varphi_2] \times [\theta_1, \theta_2] \to S_i$$
$$\Phi_i(\varphi, \theta) = (r_i \cos\varphi \sin\theta,\ r_i \sin\varphi \sin\theta,\ r_i \cos\theta)$$

For each i, we compute

$$\partial_\varphi \Phi_i(\varphi, \theta) = (-r_i \sin\varphi \sin\theta,\ r_i \cos\varphi \sin\theta,\ 0)$$
$$\partial_\theta \Phi_i(\varphi, \theta) = (r_i \cos\varphi \cos\theta,\ r_i \sin\varphi \overset{\cos}{\cancel{\sin}}\theta,\ -r_i \sin\theta),$$

so we have

$$\partial_\varphi \Phi_i \times \partial_\theta \Phi_i = (-r_i^2 \cos\varphi \sin^2\theta,\ -r_i^2 \sin\varphi \sin^2\theta,\ -r_i^2 \sin\theta\cos\theta)$$

and $\|\partial_\varphi \Phi_i \times \partial_\theta \Phi_i\| = r^2 \sin\theta$

$\boxed{\text{Problem 3-1}}$ Let X_i be a seq of iid random variables, w/ $E(X_i) = 0$ and $\sigma_i^2 = Var(X_i) = 1$. Then, ~~if~~ denoting $S_n = \sum_{i=1}^{n} X_i$, we have $\frac{S_n}{\sqrt{n}}$ converges to the standard normal distribution $N(0,1)$ in distribution,

$\boxed{\text{Problem 3-2}}$ ~~The~~ The CLT ~~P~~ above can be proven using more general result: For each $n \in N$, let $r_n \in N$ and $X_{n1}, \dots, X_{nr_n}$ be seq of random variables that are independent. ~~Suppose~~ Suppose that $S_n = \sum_{k=1}^{r_n} X_{jk}$.
We may further assume that $E(X_{jk}) = 0$, and we denote ~~$A_n^2 = \sum_{j=1}^{n} \sum_{k=1}^{r_n} \sum_{j=1} Var(S_n)$~~ $\sum_{j=1}^{n} var(S_j)$, and suppose the $\boxed{A_n^2 = Var(S_n)}$ condition

$$\lim_{n\to\infty} \sum_{k=1}^{r_n} \frac{1}{A_n^2} \int_{|X_{nk}| > \varepsilon A_n^2} |X_{nk}|^2 \, d\mu = 0$$

Then $S_n/\sqrt{n} \to N(0,1)$ in distribution.

~~If we set~~ If $\{X_k\}$ is a sequence of ~~random~~ iid random variables, then set $X_{nk} = X_k$ for $1 \le k \le n$ ~~$X_{nk} = X_k$~~ for each n and k, and ~~since Variance of X_{nk} is~~ the above condition is easily satisfied.

To prove this result, we want to show the characteristic functions $\varphi_{S_n}(t)$ converge to $\varphi_{N(0,1)}(t)$ $\forall t \in \mathbb{R}$.

Recall that $\varphi_{N(0,1)}(t) = 1 - \frac{t^2}{2}$. Note that

$$\left| e^{itx} - 1 - \frac{t^2}{2} \right| \leq \min \left\{ \frac{|t|^2}{2}, \frac{|t|^3}{6} \right\}, \text{ and, so}$$

$$\mathbb{E} \left| e^{it\cancel{X}} - 1 - \frac{t^2}{2} \right| \leq \mathbb{E} \min \left\{ \frac{|tX|^2}{2}, \frac{|tX|^3}{6} \right\}.$$

The RHS can be bounded above by

$$\int \frac{|tX|}{2}$$ (Some useful integral that I'm forgetting as of now!)

Ultimately, to compute φ_{S_n}, we recall that

$$\varphi_{S_n} = \prod_{j=1}^{r_n} \varphi_{X_{nj}}$$ since X_{nj} is independent.

So if we can show $\prod_{j=1}^{r_n} \left| \varphi_{X_{nj}} - 1 - \frac{t^2}{2} \right|$ is small, the continuity theorem for the weak convergence of r.v.'s will guarantee the result. $\boxed{S}$

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Problem 3-3

Let $S_n = \sum_{j=1}^{n} X_j$, X_j independent, and $\mathbb{P}(X_j = j^{-1/2}) = \mathbb{P}(X_j = -j^{-1/2}) = 1/2$. Observe that

$\mathbb{E}(X_j) = 0$ while $\mathbb{E}(|X_j|^2) = j^{-1}$. Then

$\frac{S_n}{j^{-1}\sqrt{j}} = \sqrt{j}\, S_n$ may not converge to $\mathcal{N}(0,1)$.

Of course X_j is not iid, so CLT may not apply.

Problem 3-4

$\frac{1}{S_n}$ $\qquad j^{\alpha}\sqrt{j} < j^{1/2}$

$\mathbb{E}(|X_j|^2) = j^{2\alpha}$. To apply CLT, we may have

$\alpha < -1$ to guarantee the convergence.